

Massive identification of asteroids in three-body resonances

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Abstract

An essential role in the asteroidal dynamics is played by the mean motion resonances. Two-body planet-asteroid resonances are widely known, due to the Kirkwood gaps. Besides, so-called three-body mean motion resonances exist, in which an asteroid and two planets participate. Identification of asteroids in three-body (namely, Jupiter-Saturn-asteroid) resonances was initially accomplished by D. Nesvorný and A. Morbidelli (1998), who, by means of visual analysis of the time behaviour of resonant arguments, found 255 asteroids to reside in such resonances. We develop specialized algorithms and software for massive automatic identification of asteroids in the three-body, as well as two-body, resonances of arbitrary order, by means of automatic analysis of the time behaviour of resonant arguments. In the computation of orbits, all essential perturbations are taken into account. We integrate the asteroidal orbits on the time interval of 100000 yr and identify main-belt asteroids in the three-body Jupiter-Saturn-asteroid resonances up to the 6th order inclusive, and in the two-body Jupiter-asteroid resonances up to the 9th order inclusive, in the set of ~ 250000 objects from the “Asteroids – Dynamic Site” (AstDyS) database. The percentages of resonant objects, including extrapolations for higher-order resonances, are determined. In particular, the observed fraction of pure-resonant asteroids (those exhibiting resonant libration on the whole interval of integration) in the three-body resonances up to the 6th order inclusive is $\approx 0.9\%$ of the whole set; and, using a higher-order extrapolation, the actual total fraction of pure-resonant asteroids in the three-body resonances of all orders is estimated as $\approx 1.1\%$ of the whole set.

Keywords: Asteroids; Asteroids, dynamics; Celestial mechanics; Resonances

1 Introduction

A substantial role of resonances in the dynamics of asteroids became evident with the discovery of resonant “gaps” in the asteroid belt by D. Kirkwood in

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1867. The deepest minima in the distribution of asteroids in the semimajor axes of their orbits correspond to the mean motion resonances 2/1, 3/1, 4/1, 5/2, and 7/3 with Jupiter. Mean motion resonance represents a commensurability between the mean frequencies of the orbital motions of an asteroid and a planet. Apart from the mean motion resonances, so-called secular resonances (Murray and Dermott, 1999; Morbidelli, 2002), representing commensurabilities between the precession rates of the orbits of an asteroid and a planet, are important in forming the dynamical structure of the asteroid belt.

There are two important classes of the mean motion resonances: apart from the usual (two-body) mean motion resonances of an asteroid and a planet, an appreciable role in the asteroidal dynamics is played by so-called three-body mean motion resonances (Murray et al., 1998; Nesvorný and Morbidelli, 1998, 1999; Morbidelli, 2002). In the latter case, the resonance represents a commensurability between the mean frequencies of the orbital motions of an asteroid and two planets (e.g., Jupiter and Saturn):

$$m_J \dot{\lambda}_J + m_S \dot{\lambda}_S + m \dot{\lambda} \approx 0, \quad (1)$$

where $\dot{\lambda}_J$, $\dot{\lambda}_S$, $\dot{\lambda}$ are the time derivatives of the mean longitudes of Jupiter, Saturn, and asteroid, respectively, and m_J , m_S , m are integers.

In view of the “overdensity” of the three-body resonances in the phase space of the asteroidal motion, Nesvorný and Morbidelli (1998) asserted that “the three-body mean motion resonances seem to be the main actors structuring the dynamics in the main asteroid belt”.

Chaotic behaviour, which is often present in the dynamics of celestial bodies, is usually due to interaction of resonances (as in any Hamiltonian system, see Chirikov 1979), but not always it is known which are the interacting resonances that give rise to chaos. It is especially difficult to identify three-body resonances. How to distinguish between resonant and non-resonant motions? To solve this problem, a “resonant argument” (synonymously “resonant phase” or “critical argument”) is introduced. It is a linear combination of some angular variables of a system under consideration; in the planar asteroidal problem it is given by

$$\sigma_{p_J, p_S, p} = m_J \lambda_J + m_S \lambda_S + m \lambda + p_J \varpi_J + p_S \varpi_S + p \varpi, \quad (2)$$

where λ_J , λ_S , λ , ϖ_J , ϖ_S , ϖ are the mean longitudes and longitudes of perihelia of Jupiter, Saturn, and an asteroid, respectively, and m_J , m_S , m , p_J , p_S , p are integers satisfying the D’Alembert rule (Morbidelli, 2002):

$$m_J + m_S + m + p_J + p_S + p = 0. \quad (3)$$

If resonant argument (2) librates (similarly to librations of a pendulum), the system is in resonance; if it circulates, the system is out of resonance. The motion of the system at the border between librations and rotations corresponds to the *separatrix*. Thus the pendulum dynamics provides a graphical model of resonance. In a certain sense this model of resonance is “universal” (Chirikov, 1979). In particular, the motion in three-body resonances can be described in

the perturbed pendulum model (Murray et al., 1998; Nesvorný and Morbidelli, 1998, 1999; Shevchenko, 2007).

An important parameter of a mean motion resonance is its *order* q , equal to the absolute value of the algebraic sum of the coefficients at the mean longitudes in the resonant argument:

$$q = |m_J + m_S + m|. \quad (4)$$

The resonant order q is important, because it is the power in which the eccentricity is raised in the coefficient of the leading resonant term in the expansion of the perturbing function (Nesvorný and Morbidelli, 1998). The corresponding subresonance width (characterizing also its “strength”) is proportional to the square root of this coefficient. Thus the value of q determines this important property of the leading subresonance.

In the case of two-body resonances, the role of the resonant order q (defined below in Section 6) is analogous: the coefficient of the leading resonant term is proportional to e^q (Nesvorný and Morbidelli, 1998), where e is the asteroidal eccentricity.

However note that, when there is no strong overlapping of subresonances, the resonant order q is not related to the width of the whole resonant multiplet, because the separation of subresonances depends solely on the secular precession rates of the pericentres (Nesvorný and Morbidelli, 1999); thus the degree of overlap (and hence, chaos) in the multiplets is expected to asymptotically decrease with the resonant order (Nesvorný and Morbidelli, 1999; Morbidelli and Guzzo, 1997).

One may expect that, generally, broader the leading subresonance of a mean motion resonance, greater is the number of objects residing in this mean motion resonance. However, no strict correlation exists, due to a competition of various dynamical and physical processes, populating or depopulating the resonances. We shall discuss this further in more detail.

In our procedure of resonance identification, described in detail below, we limit the set of possible combinations of the integers m_J , m_S , m by adopting the following conditions:

$$q \leq q_{\max}, \quad (5)$$

$$|m_J|, |m_S|, |m| \leq M_{\max}, \quad (6)$$

where $q_{\max} = 6$ and $M_{\max} = 8$.

Nesvorný and Morbidelli (1998) used condition (5) (presumably with $q_{\max} = 10$, as follows from data in table 3 in Nesvorný and Morbidelli 1998). In (Nesvorný and Morbidelli, 1999), instead of (6), the following truncation condition was used:

$$|m_J| + |m_S| + |m| \leq Q_{\max} \quad (7)$$

(see eqs. (29) and (30) and comments on them in Nesvorný and Morbidelli 1999).

We identify the three-body resonances in the current motion of asteroids with known orbital elements. The limitations of our study are as follows: solely the asteroids in the main belt are considered (i.e., the semimajor axes are in the range from 2 to 4 AU); solely the three-body resonances with Jupiter and Saturn are taken into account; the resonances are considered in the planar problem, i.e., the longitudes of nodes in the expression for the resonant argument are ignored; the maximum considered order q_{\max} of the three-body resonances is set equal to 6.

Our project is intended for the resonance analysis of the orbital data presented at the “Asteroids – Dynamic Site” (AstDyS) maintained by A. Milani, Z. Knežević and their coworkers (<http://hamilton.dm.unipi.it/cgi-bin/astdys/>). We take the orbital data for the analysis from this database. Thus the total set under analysis contains ≈ 250000 objects.

The basic purpose of our work is to identify the current three-body resonances that all the asteroids from the given set are currently involved in. More specifically, each object from the set should be put in correspondence to a three-body resonance (or none, if there is no resonance). The first attempt of massive identification of asteroids in three-body resonances was made by Nesvorný and Morbidelli (1998): 255 objects were identified to be in three-body resonances. The libration/circulation of the resonant argument for asteroids suspected to reside in the resonances was analyzed visually. In our case the data set is much greater, and therefore the procedure ought to be completely automatic. Besides, here we apply a unified bound on the order. This allows one to construct a homogeneous identification list for a further statistical analysis.

To form a general statistical view of the resonant structure of the main belt, we also accomplish a massive identification of asteroids in two-body resonances with Jupiter, and compare the abundances of asteroids in three-body and two-body resonances.

2 The identification matrix

As a first stage of the identification process we build an “identification matrix”. It consists of two main columns. The first one contains designations of resonances, and the second one contains the corresponding resonant values of the semimajor axis.

The designations of resonances are given in the notation $m_J m_S m(q)$, where m_J , m_S , m are the integer coefficients in the resonant argument (2), and q is the resonant order, as defined above. The values of m_J , m_S , m are given with their signs. Thus, examples of this notation look like as follows: 5-2-2(1), 2+2-1(3).

We construct a set of the resonant arguments for all possible three-body resonances up to a fixed order q_{\max} in the following way.

We fix the maximum absolute value M_{\max} of each integer m_J , m_S , m to be equal to $q_{\max} + 2$, where q_{\max} is the maximum resonant order. It is assumed that

$$m_J > 0, \quad \gcd(m_J, m_S, m) = 1, \quad (8)$$

where “gcd” stays for the greatest common divisor. It is set to be equal to 1 to avoid the higher order harmonics with greater *multiplicity*. The multiplicity is defined as equal to $\gcd(m_J, m_S, m)$. The harmonics with multiplicity greater than 1 are not discernible in our identification procedure, because their arguments librate simultaneously, though with different amplitudes. (Consider, e.g., such resonances as 4–2–2 and 8–4–4. The second one, which has multiplicity equal to 2, in our procedure is set to be equivalent to the first one.)

Then we search through all possible combinations of m_J, m_S, m and identify those satisfying the D’Alembert rule (3) and our technical restrictions (5) and (6).

Let us demonstrate how the resonant value of the semimajor axis is calculated. According to the definition of the three-body resonance (1), the time derivative $\dot{\sigma}_{p_J, p_S, p}$ should be equal to zero. Let us, following Murray et al. (1998), assume that $\dot{\varpi} \approx 0$, in the first approximation. Then, for the resonant value of mean motion, one has

$$n_{\text{res}} = -\frac{1}{m}(m_J \dot{\lambda}_J + m_S \dot{\lambda}_S + p_J \dot{\varpi}_J + p_S \dot{\varpi}_S). \quad (9)$$

Using Kepler’s third law, one obtains for the resonant semimajor axis

$$a_{\text{res}} = \left(\frac{k}{n_{\text{res}}} \right)^{2/3}, \quad (10)$$

where n_{res} is given by formula (9), and k is the Gauss constant.

Holman and Murray (1996) and Murray et al. (1998) obtained an approximate formula for the precession rate of an asteroid’s orbit:

$$\dot{\varpi} \approx \frac{\mu}{2\pi} \left(\frac{a}{a_J} \right)^{1/2} \varepsilon^2 n_J, \quad (11)$$

where $\mu \approx 1/1047$ is the mass of Jupiter in units of the mass of the Sun, n_J and a_J are Jupiter’s mean motion and the semimajor axis of Jupiter’s orbit, respectively, and

$$\varepsilon = \frac{a_J - a}{a_J}. \quad (12)$$

For a , we substitute here the resonant value of the semimajor axis as given by Eq. (10). Thus the value of $\dot{\varpi}$ is calculated. Iterating, one obtains an adequate value of a_{res} . The second primary column of the identification matrix is filled with the resonant values of the semimajor axes, calculated as described, with the accuracy of no less than 10^{-3} AU. This accuracy far exceeds the necessary one, because we check the asteroids for belonging to a given resonance in a far greater neighborhood ($\pm 10^{-2}$ AU) of the computed resonant value of a_{res} .

Nesvorný and Morbidelli (1999) calculated a_{res} of the leading subresonances (i.e., of the multiplet components $\sigma_{0,0,-m_J-m_S-m}$) of 19 three-body resonances

with the accuracy of 10^{-4} AU: they equated the time derivative of Eq. (2) to zero, and, using the values of $\dot{\lambda}_J$ and $\dot{\lambda}_S$ as given by Bretagnon (1990) and ϖ as found using the code by Milani and Knežević (1994), calculated $\dot{\lambda}$. All values of a_{res} given in (Nesvorný and Morbidelli, 1999, table 1) agree quite closely with the corresponding values of a_{res} calculated here iteratively, as described above, for our matrix. The agreement is illustrated in Table 1, where an extract from the identification matrix is presented.

Table 1: An extract from the identification matrix

m_J	m_S	m	q	a_{res} (AU) (this study)	a_{res} (AU) (Nesvorný and Morbidelli, 1999)
2	3	-1	4	2.3912	—
4	-2	-1	1	2.3978	2.3977
2	2	-1	3	2.6148	2.6155
4	-3	-1	0	2.6232	2.6229
1	4	-1	4	2.7432	—
3	-1	-1	1	2.7527	2.7525
1	3	-1	4	3.0673	—
3	-2	-1	0	3.0798	3.0790
5	-7	-1	3	3.0925	—
5	-2	-2	1	3.1744	3.1751

3 Dynamical identification

We use the following procedure of dynamical identification.

First of all, each asteroid’s orbit from the adopted set of 249567 objects is computed for 10^5 yr. The perturbations from all planets (from Mercury to Neptune) and Pluto are taken into account. The hybrid integrator of *mercury6* package (Chambers, 1999) is used. In some cases the *orbit9* integrator (<http://adams.dm.unipi.it/~orbmain/orbfit/>) is used as well, to verify the results. The computed trajectories are kept in files for further usage. The trajectories are output with the time step of 1 yr.

After integration is over, the objects are taken from the adopted set, and the mean value of the semimajor axis is computed for each object. Using this value, a set of preliminary resonant arguments σ_{res} is found in the identification matrix. Each argument σ_{res} is then analyzed on the presence of libration/circulation, using the computed trajectory of the object.

We distinguish two types of resonant libration: pure and transient. By definition, the libration is pure, if it lasts during the whole time interval of integration, i.e., 10^5 yr. An example of such libration is given in Fig. 1, where the time behaviour of the resonant argument, alongside with the orbital elements, is demonstrated for asteroid 463 Lola, resonance $4 - 2 - 1$.

The libration is defined as transient, if circulation appears at any time during this interval. An example is given in Fig. 2; this is the case of asteroid 490 Veritas, resonance $5 - 2 - 2$. In Fig. 2, it may seem unusual that Veritas exhibits circulation of the resonant argument while the semimajor axis remains almost constant (especially in comparison with Fig. 9, which is considered below and which shows the orbital elements and resonant argument of 1915 Quetzálcoatl, residing in the two-body transient resonance $3/1$). This difference is explained by a large difference in the widths of the resonances: e.g., at $e = 0.1$ the width of resonance $3/1$ is ~ 5 times greater than that of resonance $5 - 2 - 2$ (see fig. 1 in Morbidelli and Nesvorný 1999). Moreover, 1915 Quetzálcoatl has a very large eccentricity ($\sim 0.6-0.8$). At $e = 0.1$, the half-width of the $5 - 2 - 2$ resonance is ≈ 0.002 AU (see table 1 in Nesvorný and Morbidelli 1999 and fig. 1 in Morbidelli and Nesvorný 1999). For Veritas, the eccentricity is ~ 0.06 (see Fig. 2), hence the half-width is even smaller. This makes the shift in a , when σ is circulating, almost imperceptible, especially in the digitally unfiltered a . When a is filtered, such shifts look more obvious; see fig. 2 in (Knežević et al., 2002) and fig. 3 in (Tsiganis et al., 2007). Note that, when the circulations are short-term, the shifts in a can be imperceptible even in the digitally filtered element: see fig. 1 in (Nesvorný and Morbidelli, 1998), where the time behaviour of Veritas in the digitally filtered elements is shown.

To distinguish between transient-resonant and non-resonant behaviours we introduce a technical parameter: the *resonance minimum time*, which we set to be equal to 20000 yr. Roughly speaking, this is the minimum time interval of resonant librations, for an asteroid to be regarded as resonant. An exact definition is given below.

Nesvorný and Morbidelli (1998) identified resonant behaviour visually and used more subjective criteria, according to which the asteroid is resonant if “(1) the corresponding resonant angle shows evident librations during the integration time span or (2) the resonant angle circulates with a period longer than several thousand years” (Nesvorný and Morbidelli, 1998). In fact the second criterion threshold is analogous to our resonance minimum time, because our automatic procedure measures the time intervals of libration as such when no circulation is present, as described in detail below. Our criterion threshold, being larger, is more restrictive; however, in both cases the threshold is much greater than the timescales of circulation far from resonance; the latter timescales are of the order of asteroidal orbital periods.

The complete procedure of identification is as following. When computing each trajectory, the resonant argument value for the guiding subresonance is calculated at each step in time, and these values are written in a file, in function of time. After the trajectory computation is over, the time behaviour of the resonant argument is analyzed. If a current value of the resonant argument is different from its previous value by more than 2π , this means that there is a break, which takes place if there is either circulation or apocentric libration. To distinguish between these two cases, the same procedure is repeated, but with an artificial shift of the resonant argument. This shift is equal to π . Then, if there is apocentric libration, it turns into pericentric one, and there are no breaks.

At the beginning of the trajectory analysis, two variables are initiated: the first one is the duration of current libration, and the second one is the total time of libration. If circulation whenever starts, the first variable value is added to the second one and the first variable is reset to zero. At the end of the analysis, it is checked whether the second variable (total time of libration) is equal to the full time of computation (10^5 yr). If yes, then the asteroid is regarded to be in *pure resonance*. If no, but the second variable value exceeds the *resonance minimum time*, then the asteroid is in *transient resonance*. If the second variable value is less than the *resonance minimum time*, then the asteroid is regarded to be non-resonant. The adopted pure/transient division of resonances is thus mostly technical, because it depends on the chosen time interval of computation.

We use solely the direct method of identification, i.e., the resonant argument is checked on the subject of libration. Any secondary and/or auxiliary criteria, such as the semimajor axis behaviour, the location of objects in the proper “semimajor axis – eccentricity” plane, the Lyapunov exponents are left for future explorations, — with a partial exception in the latter case, see Section 5.

In our identification procedure, it is formally possible that an asteroid might be identified as belonging to two or even more resonances, because all resonances in a rather broad neighbourhood in semimajor axis ($\pm 10^{-2}$ AU, as mentioned above) in the identification matrix are scanned on the subject of libration. If two three-body resonances are close enough in semimajor axis, and especially if they also overlap with a strong two-body resonance, an asteroid may intermittently diffuse from one to another resonance; a small number of such objects have been identified, as belonging to two resonances. In our lists, each object of this kind has been finally attributed to the resonance where it stayed for a longer time, to avoid complication of statistics. Analysis of such objects (we call them “rogue-resonant” asteroids) will be given elsewhere.

Besides, since the data used is unfiltered, some misidentifications may take place when the libration amplitude is high. To estimate the formal statistical error of our procedure, we randomly chose 300 asteroids identified as pure-resonant and visually checked the libration of their resonant arguments. It turned out that only 4 objects (1.3% of the set) were misidentified. Thus the statistical error of the procedure is $\approx 1\%$.

All identified pure three-body-resonant asteroids, grouped according to association to a given resonance, are listed in the Appendix A. The top ten resonances, that are most “populated”, are listed in Table 2. The last column of Table 2 contains analytical estimates of the resonance width (at the asteroidal eccentricity $e = 0.1$), according to (Nesvorný and Morbidelli, 1999, table 1). One can see that, generally, broader the resonance, greater is the number of objects residing in it. However, this tendency is not strict. The reason is that the dynamics here is strongly interrelated with physics: e.g., a collisional disintegration of an asteroid can strongly increase abundance of objects in a particular resonance, thus disturbing the expected correlation.

For each resonance, statistics on the asteroids in pure and transient librations have been calculated. The statistical results are summarized in Table 3. The fraction of asteroids in three-body resonances (transient plus pure) turns out to

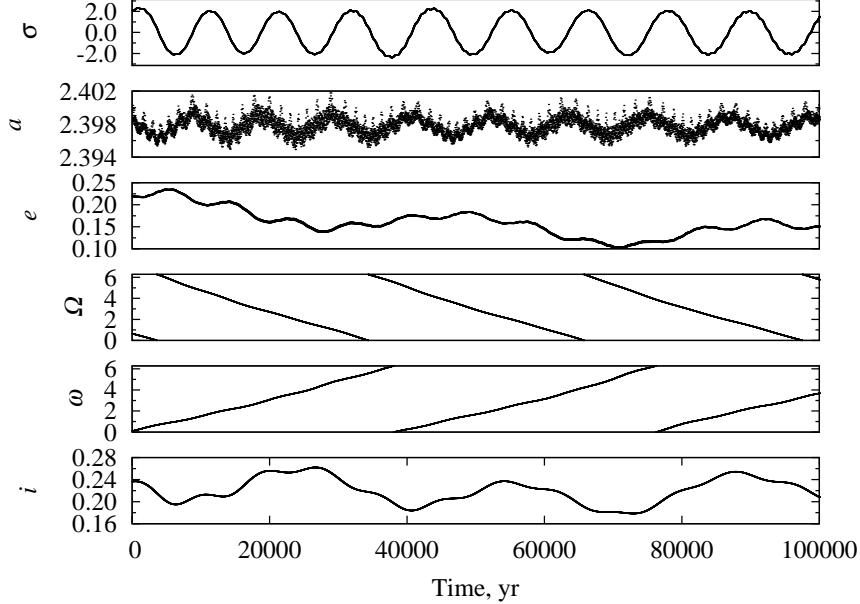


Figure 1: The orbital elements and resonant argument of 463 Lola. Resonance $4 - 2 - 1$.

be $\approx 4.4\%$ of the whole set. This is rather close to the value 4.6% (≈ 1500 in the set of ≈ 32400) found by Nesvorný and Morbidelli (1998) to serve as a lower bound for the relative number of resonant asteroids. As follows from Table 3, the fraction of asteroids in pure three-body resonances turns out to be $\approx 0.94\%$ of the whole set.

The third column of Table 3 contains a prediction for the numbers of asteroids residing in resonances of all orders; this subject is discussed in the next Section.

Let us consider the location of three-body-resonant asteroids in the “semi-major axis — eccentricity” plane. For each asteroid in a given resonance we calculate the average values (over the whole time interval of integration) of the semimajor axis and eccentricity and plot these values in the “ $a-e$ ” plane. We have accomplished this procedure for each resonance with known structure of the resonant multiplet, as calculated by Nesvorný and Morbidelli (1999), so that the separatrix of the leading subresonance could be drawn. An example of such a plot is given in Fig. 3. It is clear that the percentage of “outliers” (objects out of the separatrix cell) is zero. This confirms the good accuracy of the accomplished identification procedure in the case of this particular resonance. A detailed study of the “ $a-e$ ” plots will be given elsewhere.

We have cross-checked the lists of resonant objects identified in our study and in (Nesvorný and Morbidelli, 1998). It turns out that the number of resonant

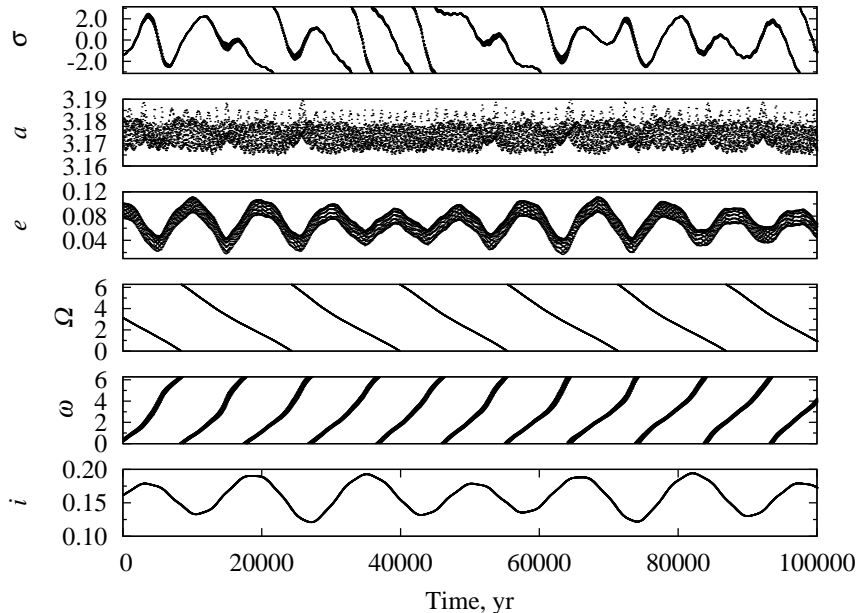


Figure 2: The orbital elements and resonant argument of 490 Veritas. Resonance $5 - 2 - 2$.

objects listed by Nesvorný and Morbidelli (1998) but not identified as resonant in our study does not exceed 1% of the list of Nesvorný and Morbidelli (1998). This confirms that the differences in the methodologies of identification in the two studies do not play any significant role in what concerns the reliability of results.

4 Expected abundances of asteroids in high-order three-body resonances

The obtained list of resonant asteroids is obviously not complete, due to the limitations of identification criteria. First of all, the resonant order $q \leq 6$. Presumably, there is a lot of objects in resonances of higher orders. To take account of them, let us analyze the asteroid distribution in the resonant order q . The necessary data as derived from the results of the identification procedure are presented in Table 4.

The constructed differential distributions (histograms) in q are shown in Fig. 4 for the case of transient plus pure resonances, and in Fig. 5 for the case of pure resonances.

As an approximating function, we have chosen the power law

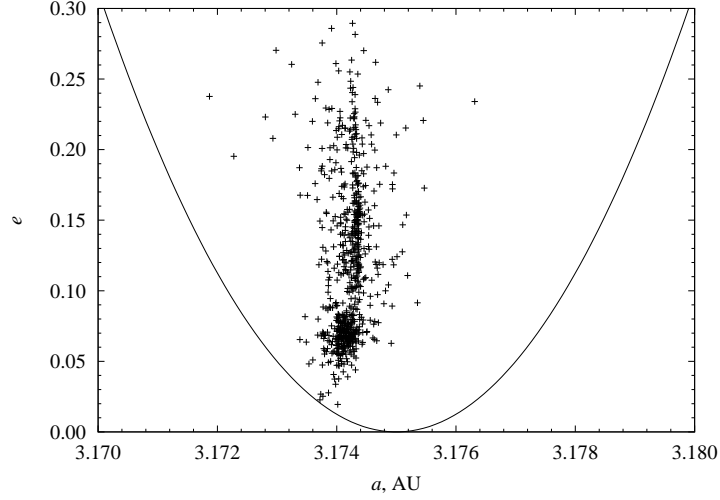


Figure 3: The asteroids identified to be in resonance $5 - 2 - 2$: location in the “ $a-e$ ” plane. Solid curve: the separatrix of the leading subresonance, as calculated in (Nesvorný and Morbidelli, 1999).

Table 2: The top ten most populated three-body resonances Sun–Jupiter–asteroid

m_J	m_S	m	a_{res} (AU)	Trans. + pure, number of objects	Pure, number of objects	Res. width (AU) at $e = 0.1$ (Nesvorný and Morbidelli, 1999)
5	-2	-2	3.1744	699	182	0.0056
4	-2	-1	2.3978	688	595	0.0024
3	-2	-1	3.0798	621	134	0.0045
3	-1	-1	2.7527	540	203	0.0019
2	2	-1	2.6148	470	34	0.00015
4	-3	-1	2.6232	455	90	0.0009
2	3	-1	2.3912	343	56	–
5	-7	-1	3.0925	314	48	–
1	3	-1	3.0673	300	52	–
1	4	-1	2.7432	284	47	–

$$N = aq^b, \quad (13)$$

where a and b are two fitting parameters; $b < 0$. We have also tried the exponential law $N \propto \exp(cq)$ (where $c < 0$), but it has turned out to be inappropriate, the statistical significance of fitting being very low.

Since we are interested in the tail behaviour of the distributions, we have

Table 3: Asteroids in three-body resonances, statistics

	NM98*	This study, $q_{\max} = 6$	This study, prediction for $q_{\max} = \infty$
The whole set of objects	5400	249567	249567
Objects with integrated orbits	836	249567	249567
Transient+pure-resonant objects	255	11039	—
The same, fraction of the studied set	4.7%	4.4%	—
Pure-resonant objects	—	2338	2854
The same, fraction of the studied set	—	0.94%	1.1%

*Nesvorný and Morbidelli (1998).

Table 4: Abundances of asteroids in three-body resonances in function of q

q	0	1	2	3	4	5	6
Transient+pure	1371	2884	1367	2256	1394	997	770
Pure	356	1119	194	251	214	117	87

used the data for $q \geq 1$, ignoring the specific case $q = 0$. In the case of all resonant (transient-resonant plus pure-resonant) asteroids (see Fig. 4) we find $a = 2821 \pm 454$, $b = -0.57 \pm 0.18$; and the correlation coefficient $R^2 = 0.71$. As soon as $|b| < 1$, the predicted number of asteroids in high-order resonances is formally infinite; in practice this means that it can comprise up to $\sim 100\%$ of the whole set.

In the case of pure-resonant asteroids (see Fig. 5) we find $a = 1095 \pm 103$, $b = -1.64 \pm 0.28$; $R^2 = 0.94$. As soon as $|b| > 1$, the predicted total number of asteroids in high-order resonances is finite. Using Eq. (13), the number of objects with $q \geq 7$ in the studied set is estimated to be equal to ≈ 516 , or $\approx 22.1\%$ of the identified number (2338). Therefore, the predicted total number of asteroids in pure three-body resonances is estimated as 2854, constituting 1.1% of the whole set (249567).

A note of caution is in order here. One has to admit that the fits made in this Section are based on few points, and, therefore, any statistical predictions, made with these formulas, are uncertain. What is more, one cannot even expect to find smooth distributions and/or strict correlations in this field of research, where dynamics is strongly interrelated with physics: the abundances of objects in resonant groups are regulated by various processes, e.g., by collisions and the Yarkovsky effect. Interactions with asteroidal families are important. Of course, higher order resonances should be directly analyzed in the future.

Another complicating factor is that the subresonance overlap (and hence, the degree of chaoticity) in the multiplets is expected to decrease asymptotically at $q \gg 1$. The reason is that the subresonance typical width scales with

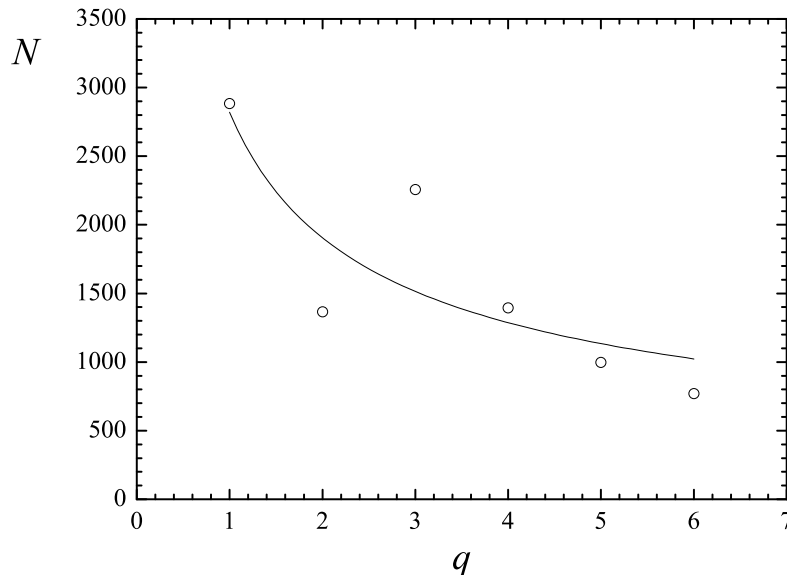


Figure 4: Circles: the distribution of all resonant (transient-resonant plus pure-resonant) asteroids in resonant order $q \geq 1$. Solid curve: the power-law fitting.

q as $\sim e^q$ (where e is the eccentricity), whereas the subresonances separation (determining the multiplet width) remains basically constant, depending solely on the secular precession rates of the pericentres; thus the overlap/interaction of subresonances in the multiplets decreases asymptotically at $q \gg 1$ (Nesvorný and Morbidelli, 1999, p. 268–269). When the ratio of the subresonances separation to the width of the leading subresonance is much greater than one, the separatrix chaotic layers are exponentially thin with this ratio, see (Chirikov, 1979; Shevchenko, 2008, 2011). It is straightforward to suppose that, if one fixes the asteroidal eccentricity, three-body resonances with increasing order become basically regular (Nesvorný and Morbidelli, 1999). Thus, if the resonances are populated uniformly in the eccentricity, the power-law extrapolation for the high-order transient-resonant populations fails beyond some order. Only if this critical order is high enough, the extrapolation-based estimates might be appropriate.

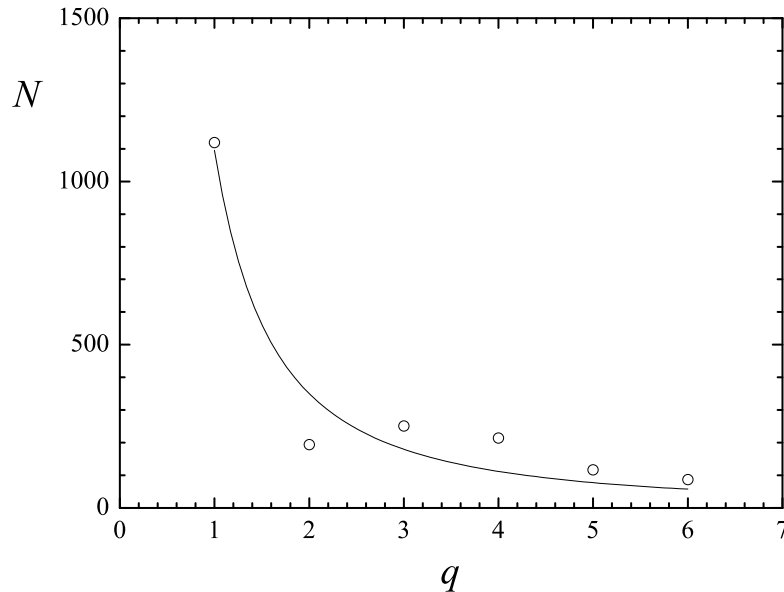


Figure 5: Circles: the distribution of pure-resonant asteroids in resonant order $q \geq 1$. Solid curve: the power-law fitting.

5 Lyapunov exponents in three-body resonances

The AstDyS database provides information on the maximum Lyapunov exponents for almost all asteroids contained in it. The provided values are computed on time intervals of 2 mln yr, i.e., on time intervals 20 times greater than that we use in our identification procedure. It is instructive to check how the data on Lyapunov exponents correlate with the pure/transient division (adopted in our study) of the resonant asteroids.

In the transient case the motion is expected to be chaotic, and in the pure case to be essentially regular, because transitions from libration to circulation and vice versa are inevitable in a separatrix chaotic layer of resonance. However, the adopted relatively short integration time may imply that many “pure”-resonant asteroids become transient on longer timescales.

First of all, let us build distributions of resonant asteroids in L (the maximum Lyapunov exponent). The constructed differential distributions (histograms) are presented in a single plot in Fig. 6. The Lyapunov exponents are given in units of $(\text{mln yr})^{-1}$. (Note that in the AstDyS database they are given in

units of yr^{-1} .) N is the number of objects in the interval $(L, L + \Delta L)$, where $\Delta L = 10$. The grey histogram is for the asteroids in transient resonances, and the black one is for the asteroids in pure resonances. The histograms are cut off at $L = 500$, because the objects with greater values of L are rare; they are all transients.

From Fig. 6 it is clear that the transients have a much more extended distribution in L , in comparison with the pure-resonant objects, as expected. This difference is uniform in all resonant groups, as can be directly seen from Fig. 7, where the maximum Lyapunov exponents are plotted versus the semimajor axis a : the positions of open dots (representing transients) extend to much greater heights, in comparison with the pure-resonant objects, in all resonant groups. Note that this plot is built without any cut-off in L , i.e., all objects are shown. It is evident that only transients have very large Lyapunov exponents, corresponding to Lyapunov times as small as ≈ 570 yr.

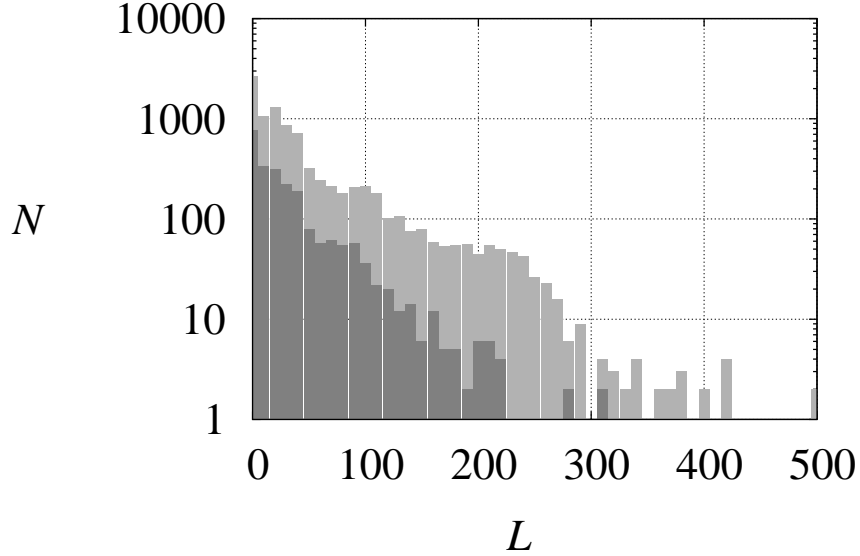


Figure 6: The distributions of resonant asteroids in the value of maximum Lyapunov exponent. Grey: asteroids in transient resonances. Black: asteroids in pure resonances.

Among all identified pure-resonant objects, the average L is $34.3 (\text{mln yr})^{-1}$, and among all identified transients it is $49.7 (\text{mln yr})^{-1}$. The Lyapunov times are ≈ 29200 and ≈ 20100 yr, respectively. Among all objects of the AstDyS database with measured Lyapunov exponents, the average $L = 20.8 (\text{mln yr})^{-1}$, corresponding to the Lyapunov time ≈ 48100 yr. Thus both pure-resonant and transient-resonant objects turn out to be more chaotic than a typical asteroid.

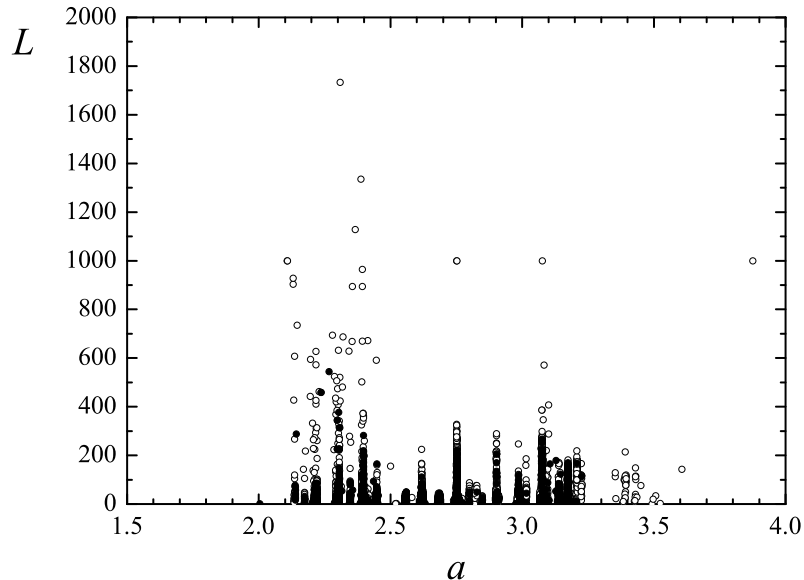


Figure 7: The maximum Lyapunov exponents versus the semimajor axis a . Open dots: asteroids in transient resonances. Black dots: asteroids in pure resonances.

The fact that many pure-resonant asteroids have definitely non-zero Lyapunov exponents signifies that they become transient on the timescales longer than the adopted integration time (10^5 yr). This underlines the conditional character of the “pure/transient” technical classification adopted in our study.

6 Two-body resonances with Jupiter

To form a more general statistical view of the actual resonant structure of the main belt, it is instructive to compare the abundances of asteroids in three-body resonances with the abundances of asteroids in two-body resonances. For this purpose, we have performed a procedure of identification of asteroids in two-body resonances with Jupiter, analogous to that described above for the case of three-body resonances. In this identification procedure, we assume circular and zero inclination orbits of perturbing planets.

Taking into account the D’Alembert rule, the resonant argument for the resonance of order q is defined by the following formula (Murray and Dermott, 1999; Morbidelli, 2002; Gallardo, 2006):

$$\sigma = (p + q)\lambda_J - p\lambda - q\varpi, \quad (14)$$

where λ_J and λ are the mean longitudes of Jupiter and an asteroid, respectively, and ϖ is the longitude of perihelion of the asteroid; q is the resonant order, p is integer.

Of course, this setting of the identification problem is a rather simplified one. When real resonant asteroids are considered, the “circular” subresonance term is not expected to dominate universally; what is more, the resonant asteroid might reside in another component of a multiplet (see Holman and Murray 1996; Murray and Holman 1997). We regard the adopted approach as a first approximation, and leave considerations of all possible subresonances for the future.

The resonant value of the semimajor axis of an asteroidal orbit is given by

$$a_{\text{res}} \approx a_J(1 + \mu)^{-1/3} \left(\frac{p}{p + q} \right)^{2/3}, \quad (15)$$

where a_J is the semimajor axis of Jupiter’s orbit, and μ is the mass of Jupiter in units of the mass of the Sun (see, e.g., Murray and Dermott 1999; Gallardo 2006).

The identification matrix is constructed in the way analogous to that described in Section 2. Similar to the case of three-body resonances, it is assumed that $\gcd(p, q) = 1$. Besides, $1 \leq p \leq 11$.

The resonant order range is limited to $0 \leq q \leq 9$. Recall that in the case of three-body resonances we have set $0 \leq q \leq 6$. Choosing the upper bounds with the difference equal to 3 allows one to adequately compare identification statistics for two-body and three-body resonances, because “a three-body resonance of a given order q should have roughly the same strength as a usual resonance of order $q + 3$ for eccentricity of about 0.05–0.10” (Nesvorný and Morbidelli, 1998). Such a trick was used in Nesvorný and Morbidelli (1998) for similar purposes.

Figs. 8 and 9 show the time behaviour of the resonant argument and orbital elements of 1915 Quetzálcoatl and 190 Ismene, as two typical examples. The top ten most “populated” resonances, identified in our study, are listed in Table 5. All identified pure two-body-resonant asteroids, grouped according to association to a given resonance, are listed in the Appendix B.¹

The resulting statistics of identified objects are listed in Tables 6 and 7. As follows from Table 6, a half of all identified asteroids in pure two-body resonances are Trojans ($1669/3132 \approx 53\%$). Pure Trojans plus pure Hildas constitute $\approx 85\%$ of all asteroids in pure two-body resonances in our set. In Table 7, the q dependence of the resonant asteroid abundances is presented. The dependence is obviously irregular and does not permit any smooth decay approximation. Especially one should point out the negligible asteroid abundances at $q = 3$ and $q = 9$.

¹One may wonder why 279 Thule is absent in the 4/3 entry. The matter is that our automatic procedure identifies it as being in transient resonance, not in pure one, because the resonant argument sometimes goes out of the range $(-\pi, +\pi)$.

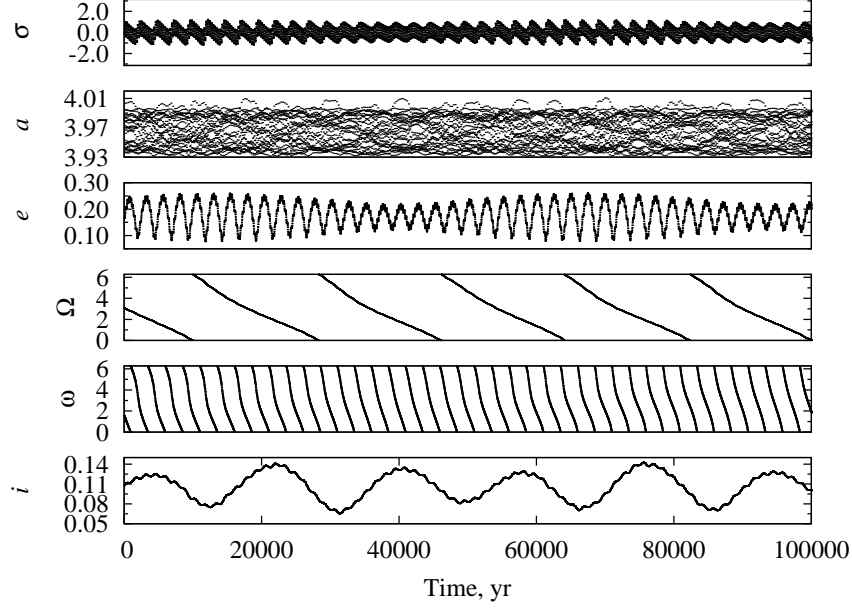


Figure 8: The orbital elements and resonant argument of 190 Ismene. Pure resonance $3/2$.

Table 5: The top ten most populated two-body resonances with Jupiter. The main belt and Trojans

$p + q$	p	q	a_{res} (AU)	Transient + pure	Pure	Notes
1	1	0	5.2043	1670	1669	Trojan swarms
3	2	1	3.9716	1033	1007	Hilda group
11	5	6	3.0766	351	49	Griqua family
2	1	1	3.2785	274	227	
8	3	5	2.7063	232	20	
7	2	5	2.2576	208	20	
10	3	7	2.3322	165	15	
11	4	7	2.6514	165	0	Alinda family
9	4	5	3.0309	163	55	
3	1	2	2.5020	63	32	
7	3	4	2.9583	51	17	

It turns out that in transient plus pure resonances the asteroids are ≈ 2.5 times more abundant in three-body resonances than in two-body resonances (11039 versus 4450); and in pure resonances the abundances are in the ratio

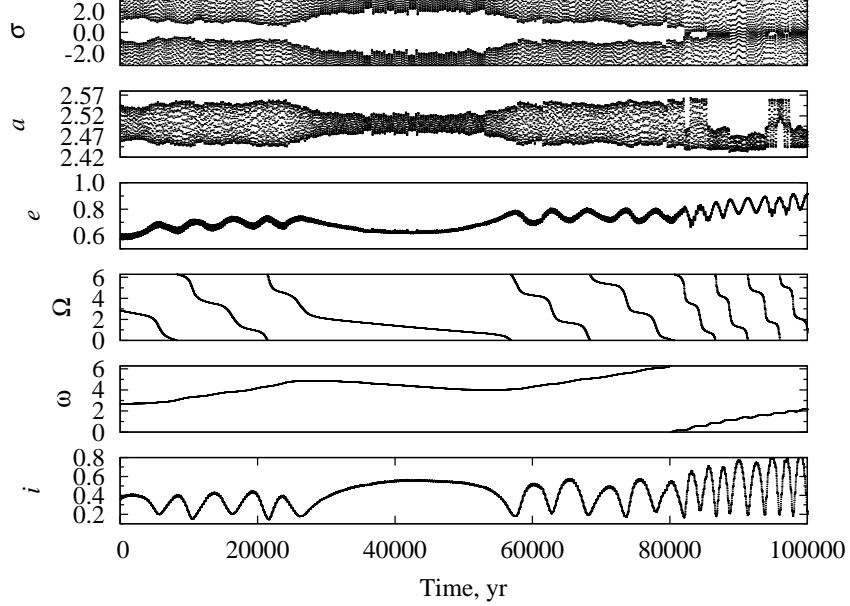


Figure 9: The orbital elements and resonant argument of 1915 Quetzálcoatl. Transient resonance 3/1.

Table 6: Asteroids in two-body resonances with Jupiter, statistics. The main belt and Trojans

	Number of resonances	Number of objects
Transient + pure	21	4450 (1.78%)
Pure	16	3132 (1.25%)

Table 7: Abundances of asteroids in two-body resonances in function of q

q	0	1	2	3	4	5	6	7	8	9
Transient + pure	1670	1310	63	15	57	603	352	342	37	1
Pure	1669	1236	32	5	19	95	49	23	4	0

$\approx 3 : 4$ (2338 versus 3132). However, if one excludes Trojans and Hildas, the abundance of three-body-resonant asteroids becomes overwhelming: in the case of transient plus pure resonances the ratio of observed abundances of three-body-resonant objects and two-body-resonant objects becomes equal to $11039/1747 \approx 6.3$; and in the case of pure resonances the ratio becomes equal to $2338/456 \approx$

5.1.

If Trojans and Hildas are not excluded, the fraction of pure-resonant asteroids among the transient+pure-resonant asteroids is ≈ 3.3 times greater in the two-body case than in the three-body one ($\approx 70.4\%$ versus $\approx 21.2\%$); thus it might seem that the two-body resonances are much more “pure” on average. However, if one excludes Trojans and Hildas, the fractions become comparable: 26.1% versus 21.2% .

One should outline that the comparative analysis given here is merely statistical and, thus, formal. However, such an analysis might provide a necessary preliminary stage for a deeper study of the comparative role of two-body and three-body resonances; such a study should comprise consideration of the physical (collisions and the Yarkovsky effect) and dynamical (transport and diffusion) processes, which lead to populating or depopulating the resonances.

7 Conclusions

1. We have identified the resonant objects (the objects residing in three-body resonances with Jupiter and Saturn in the main asteroid belt) in the set of all numbered asteroids in the AstDyS database. This set comprises 249567 asteroids catalogued up to the date of April, 2011. The list of all asteroids identified as residing in pure three-body resonances is given in Appendix A.
2. The fraction of asteroids in three-body resonances (transient plus pure) up to the 6th order inclusive turns out to be $\approx 4.4\%$ of the total studied set of 249567 asteroids. The fraction of asteroids in pure three-body resonances of the same orders turns out to be $\approx 0.94\%$ of the total studied set.
3. The top three most populated three-body resonances are: 5 -2 -2 (containing 699 transient+pure-resonant asteroids), 4 -2 -1 (688 transient+pure-resonant asteroids), 3 -2 -1 (621 transient+pure-resonant asteroids). For the pure-resonant asteroids, the “top three” resonances are: 4 -2 -1, 3 -1 -1, and 5 -2 -2, containing 595, 203, and 182 objects, respectively.
4. Using a high-order extrapolation (in the form of a power law) of the q dependence of the number of identified resonant objects, the actual total fraction of asteroids in pure three-body resonances of all orders is estimated as $\approx 1.1\%$ of the whole set. In what concerns the case of transient three-body resonances, the situation is much less certain, because the power-law extrapolation diverges.
5. We have also identified all objects residing in two-body resonances (of order $0 \leq q \leq 9$) with Jupiter in the main asteroid belt, taking the same database of asteroids. The list of all asteroids identified as residing in pure two-body resonances is given in Appendix B.

6. The half of all identified asteroids in pure two-body resonances are Trojans ($\approx 53\%$). The pure Trojans plus pure Hildas constitute $\approx 85\%$ of all asteroids in the pure two-body resonances. The q dependence of the two-body resonant abundances is clearly irregular and does not permit any smooth decay approximation. Especially one should point out the negligible asteroidal abundances at $q = 3$ and $q = 9$.
7. In the transient plus pure resonances, the identified asteroids are ≈ 2.5 times more abundant in the three-body resonances than in the two-body resonances; and in the pure resonances the abundances are comparable. However, if one excludes Trojans and Hildas, the abundance of three-body-resonant asteroids becomes overwhelming. What is more, taking into account extrapolated abundances in high-order resonances may substantially increase this overwhelming domination. Thus our analysis quantitatively verifies the assertion by Nesvorný and Morbidelli (1998) that “the three-body mean motion resonances seem to be the main actors structuring the dynamics in the main asteroid belt”.
8. We would like to point out that our results confirm the general concept of Molchanov (1968, 1969) on the omnipresence of resonances in the Solar system, however at a new level of understanding of this phenomenon.

Acknowledgements

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Appendix A

Asteroids in pure three-body resonances with Jupiter and Saturn

1 3 -1 (3): 95 423 1102 3585 11456 17433 22499 27767 34493 41164 42441 46218 47896 48131 59909 62884 63667 72791 73569 74425 76056 78202 82703 91809 105839 112669 117065 117344 129360 138837 140653 141042 145390 155877 159321 161779 166668 186951 195633 195930 199688 200344 201223 201875 209429 209437 209473 210159 218079 226525 232332 232494 238387 239477 240226

1 4 -1 (4): 9835 20596 31058 36595 37004 37611 41056 41640 50350 51002 55069 57488 58002 62632 66629 67442 71024 82600 83471 84938 87318 90368 93699 97281 107076 119544 122676 126319 130159 147292 158732 160021 166433 166516 167947 168639 178642 186262 188442 192877 212621 219368 226015 232815 235341 241349 249070

1 6 -1 (6): 10482 12785 14833 16967 18960 25508 31374 33523 34780 39875 48030 53741 67456 74729 79395 82094 95170 105575 105721 130930 147022 180268 194048 199297 203374 210376 226238 230570 232392 233526 245785

2 1 -1 (2): 1149 1670 38992 67811 105476 167454 229942 241160

2 2 -1 (3): 70 194 258 839 923 995 16233 28706 37158 48104 48110 48127 48418 48532 51196 62082 67329 71407 84437 91337 93241 117209 122690 126001 133792 154671 173752 180242 186145 212288 224957 232199 238618 243865

2 3 -1 (4): 7553 10368 39930 42942 48007 48028 48039 48105 48111 48118 48120 48121 48133 48509 48511 69468 70514 72106 74316 82341 89577 90015 90354 90546 112351 113728 115283 119372 122007 125642 129963 130692 150940 153684 157179 159616 161644 162752 167044 180486 180491 180493 180494 186531 188874 189427 197995 203427 216047 220119 222186 236300 239888 243110 244652 248126

2 4 -1 (5): 9945 12169 15628 32302 32898 33946 38779 38993 39176 48014 48033 48103 48107 59305 75243 90851 99504 120401 138965 143630 168287 180218 211512 217139 218091 239263

3 -2 -1 (0): 9864 10173 10926 17320 22785 26986 29849 29869 34592 36671 40940 41403 45312 47919 54497 56883 58847 58914 62989 64773 71551 73469 73570 73573 73798 73952 75981 76535 78756 82905 83011 83055 83103 83698 83972 84448 84640 90097 94202 97229 106316 112360 112384 112757 113085 113337 113402 113904 115445 123903 124012 127061 128255 128304

129408 130325 131350 132895 132912 133447 133714 135218 136290 138750
140372 140655 143617 144889 152112 157195 158147 160008 160150 164493
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201338 201368 202406 203309 203815 204167 207764 209436 209446 209457
209458 209482 209487 209488 210167 210978 214804 216414 216642 217332
218433 221358 221408 223303 226565 227722 228301 231654 234751 234855
234957 235444 235958 240046 243227 243296 246676 247045 248205 249330

3 -1 -1 (1): 564 947 1427 2042 2465 3534 4106 4426 5701 8068 9447
13085 13357 14169 14368 18302 19901 20118 20694 22146 22562 24011 24032
24190 25314 25891 27827 31192 33071 34104 35926 37025 41157 41868 42743
44362 44458 45486 48038 48539 50298 50954 54089 54405 54429 55893 56860
57161 57879 59296 59681 60160 61974 61980 62518 63423 64937 65543 67013
67028 67349 69300 69375 71281 71401 71423 71428 74215 74225 74314 76226
76238 76387 78768 79443 81841 81883 81927 83388 83935 87749 87893 88110
88173 89354 90647 93212 93749 95486 96994 97230 112181 114698 115643
117401 117668 118375 118835 120109 122501 122720 123213 123891 126305
127119 127499 131131 135585 138054 138096 139828 139925 140586 140605
140891 143738 146483 146507 147367 147370 148921 148949 148965 149535
152057 156621 157792 159133 161375 162113 164762 170012 170043 170655
171450 172327 172703 172908 174418 177037 179311 180571 180686 181134
181576 181761 181872 182146 185159 185934 188312 192832 195353 196624
198484 198567 198813 199291 200275 201564 201633 201847 202295 203717
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234345 235689 239615 241729 242054 244969 245373 245400 247051

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119371 125682 129748 129751 131646 134132 137237 139138 145918 153401
162608 165365 176110 177795 179798 180412 197692 205613 213218 213525
217478 220043 226224 232412 235665 240796

3 3 -2 (4): 5413 10656 11653 21807 29946 41152 48515 48516 106548
106707 121883 138820 149695 182619 185797 209431 209440 209464 209469
209478 209483 209494 223389

3 4 -2 (5): 28984 32932 33778 36169 39340 42104 56894 113850 116063
167477 168326 197436 209435 209439 209443 225316

3 4 -1 (6): 209454

4 -4 -1 (1): 52667 107282 129662 157240 163464 198658

4 -3 -1 (0): 3870 7456 7530 19700 21151 22085 22686 22904 23678
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 4684 5156 5323 5747 6167 6193 6947 7133 7831 8044 8166 8772 9777 10009
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 43023 43499 44184 44224 45128 45256 45979 47257 48101 48513 48526 50597
 51399 51412 51449 51563 51586 51628 52252 52374 52897 53095 53868 54082
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 88331 88580 88743 88817 89071 89512 89515 90121 90503 90515 90615 90977
 90988 91319 92578 92605 92863 92879 93237 93238 94688 95119 95274 96372
 96442 96683 98055 98186 98211 98416 99585 99777 105258 105378 105509
 106090 106424 106862 106995 107111 112635 112731 113045 113416 113788
 114358 114381 114393 114414 115067 115148 115739 116691 117187 118096
 119087 119128 119129 119252 119322 119327 120754 122038 122121 122144
 122171 122211 122987 124608 124893 124952 124961 125305 125404 125655
 125661 126059 126235 126356 127513 127711 127725 128307 128375 129225
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194663 194706 194804 194841 195074 195127 196258 196304 196390 196471
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4 1 -2 (3): 116793

4 1 -1 (4): 5066 28441 141359 166908 209455

4 3 -2 (5): 37957 62712 81077 81561 115442 126605 170321 180727
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5 -7 -1 (3): 19282 22198 48500 58123 58525 59375 82464 82946 83299
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5 -7 5 (3): 23539 26521 30985 71319 73466 76454 96142 106325 141402
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5 -6 -1 (2): 5534 7176 14090 14941 17149 21960 29351 29871 30477
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5 -6 5 (4): 43685 219393

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5 -3 -1 (1): 7882 29998 43935 106173 107454 207395 220455 221522

5 -2 -2 (1): 490 744 755 786 818 1072 1209 1438 1701 1731 2039 2142
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5 -1 -1 (3): 97741 209474

5 1 -2 (4): 30887 47629 48023 48117 59592 73345 75496 118814 126994
148536 149752 184749

5 2 -2 (5): 3742 11237 48022 48138 48528 51225 75787 134216 147223
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5 3 -3 (5): 41188 51919 52365

5 3 -2 (6): 28572 48012 48035 63365 69144 70259 77353 107672 118218
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220992

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6 -7 5 (4): 21416 122051 180505 224163 228870

5 4 -3 (6): 204141

6 -6 -1 (1): 5684 60723 61210 77084 77140 78925 86835 92462 129754
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6 -4 -1 (1): 48001 97759 141744 157079 217839

6 -3 -2 (1): 13797 19987 107393 122771 132844 204025 209370 237930
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6 -3 -1 (2): 76800

6 -1 -2 (3): 3426 11792 20861 91347 98826 142623 148819 170636 196499 200520

6 1 -3 (4): 57861 69553 76017 78776 116376 159801 168157 171677
184224 202322 209447 209449 209451 209453 209462 209480 209484 219975
232662 238386 240451

6 1 -2 (5): 16601 23644 30162 41621 68649 89802 92860 99716 121093
132149 134702 139246 142066 150091 155432 163292 169468 188240 194745
197727 197853 211543 215096 220219 228253 230149 241162

6 2 -3 (5): 4600 41401 51861 121579 159818 184675 209434 209467
244168

7 -7 -2 (2): 15290 15731 23485 47732 113231 170857 177020 177095
181071 209450 212665 216400 236505 248433

7 -7 -1 (1): 218044

7 -6 -2 (1): 9838 29563 97767 106034 127629 140618 178715 196816
223929 248801

7 -6 -1 (0): 33865 82020

7 -5 -2 (0): 145459

7 -4 -2 (1): 789 6473 9297 12276 30274 31230 51905 59439 62081 69966 71403
78848 89247 93712 94018 96004 97368 112284 117217 126886 132654 143342
150176 155000 178973 180648 181301 183437 184816 190375 207310 210646
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238195

7 -3 -2 (2): 16296 16431 37885 46792 47602 53099 61263 72777 88472
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233905 239178

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95778 126511 153543 156313 166240 177296 180507 183679 194470 222451
225756 228498

7 -1 -2 (4): 20157 22519 27267 29032 33884 41473 48008 48027 48029
50177 51198 53766 57387 62411 72909 79157 79676 88742 95068 99518 125118
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174793 175160 175831 213528 214159 218114 240121 245749

7 1 -3 (5): 51664 57935 65732 116950 190207 223715

7 1 -2 (6): 48020 48031 68793 131749 174692

7 2 -3 (6): 63304 66880 79094 145551 219356 237227 237964 240175

7 3 -4 (6): 120291

8 -7 -2 (1): 7282 12592 31488 36675 69219 72509 77518 99201 123051
150021 157386 161417 166474 181601 182133 185382 186607 196547 204898
206018 219402 232499 236599 244538

8 -4 -3 (1): 10 468 51250 56857 113931 134378 173978 209448 209485

8 -3 -3 (2): 105866

8 -3 -2 (3): 9791 21634 105959 106040 136430 181378 200353 225201

8 -2 -3 (3): 151028 203738

8 -1 -3 (4): 19698 41253 160914 185903

8 -1 -2 (5): 181412

8 1 -4 (5): 221458

8 1 -3 (6): 4047 4553 7439 46087 96738 116854 125638 148867 152574
219335

Appendix B

Asteroids in pure two-body resonances with Jupiter

1/1 (0): 588 617 624 659 884 911 1143 1172 1208 1404 1437 1583 1647 1749
1867 1869 1870 1871 1873 2146 2148 2207 2223 2357 2363 2456 2674 2759 2797
2893 2895 2920 3063 3240 3317 3391 3451 3540 3548 3564 3596 3708 3709 3793
3794 4007 4035 4057 4060 4063 4068 4086 4138 4489 4501 4707 4708 4709 4715
4722 4754 4791 4792 4805 4827 4828 4829 4832 4833 4834 4835 4836 4867 4902
5012 5023 5025 5027 5028 5041 5119 5120 5123 5126 5130 5144 5209 5233 5244
5254 5257 5258 5259 5264 5283 5284 5285 5436 5476 5511 5637 5638 5648 5652
5907 6002 6443 6545 6997 6998 7119 7152 7214 7352 7543 7641 7815 8060 8125
8241 8317 9023 9030 9142 9430 9431 9590 9694 9712 9713 9790 9799 9807 9817
9818 9828 9857 9907 10247 10664 10989 11089 11251 11252 11273 11275 11351
11395 11396 11397 11429 11487 11488 11509 11552 11554 11663 11668 11869
11887 12052 12054 12126 12238 12649 12658 12714 12916 12917 12921 12929
12972 12973 12974 13060 13062 13181 13182 13183 13184 13185 13229 13323
13353 13362 13366 13372 13379 13383 13385 13387 13402 13463 13475 13650
13694 13780 13782 13790 13862 14235 14268 14518 14690 14707 14791 14792
15033 15094 15398 15436 15440 15442 15502 15521 15527 15529 15535 15536
15651 15663 16070 16099 16152 16428 16667 16956 16974 17171 17172 17314
17351 17365 17414 17415 17416 17417 17418 17419 17420 17421 17423 17424
17442 17492 17874 18037 18046 18054 18058 18060 18062 18063 18071 18137
18228 18263 18268 18278 18281 18282 18940 18971 19018 19020 19725 19844
19913 20144 20424 20428 20716 20720 20729 20738 20739 20947 20952 20961
21271 21284 21370 21371 21372 21595 21599 21601 21602 21900 22008 22009
22010 22012 22014 22035 22041 22042 22052 22054 22055 22056 22059 22149
22180 22203 22227 22404 22503 22808 23075 23114 23118 23119 23126 23135
23144 23152 23269 23285 23355 23382 23383 23463 23480 23549 23622 23624
23694 23706 23709 23710 23947 23958 23963 23968 23970 23987 24018 24022
25344 25347 25883 25895 25910 25911 25937 25938 26057 26486 26510 26601
26705 28958 29196 29314 29603 29976 29977 30020 30102 30498 30499 30504
30505 30506 30510 30698 30704 30705 30708 30791 30792 30793 30806 30807
30942 31037 31342 31806 31819 31820 31821 31835 32339 32356 32370 32396
32397 32434 32435 32437 32440 32451 32461 32464 32467 32471 32475 32478
32480 32482 32496 32498 32499 32501 32513 32615 32720 32726 32794 32811
33822 34298 34553 34642 34684 34746 34785 34993 35272 35276 35277 35363
35672 35673 36259 36265 36267 36268 36269 36270 36271 36425 36624 36922
37297 37299 37300 37301 37519 37572 37685 37710 37714 37715 37716 37732
37789 37790 38050 38051 38052 38257 38574 38585 38592 38594 38596 38597
38598 38599 38600 38606 38607 38609 38610 38611 38614 38615 38617 38619
38621 39229 39264 39270 39275 39278 39280 39284 39285 39286 39288 39289
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39798 40237 40262 41268 41340 41350 41353 41355 41359 41379 41417 41426
41427 42036 42114 42146 42168 42176 42179 42182 42187 42201 42230 42277
42367 42554 42555 43212 43436 43627 43706 45822 46676 47955 47962 47963

47964 47967 47969 48438 51339 51340 51344 51345 51346 51347 51348 51350
 51351 51354 51357 51359 51360 51362 51364 51365 51378 51910 51935 51958
 51962 51969 51984 51994 52273 52275 52278 52511 52567 52645 53418 53419
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 58480 58931 59049 59355 60313 60322 60328 60383 60399 60401 60421 61610
 61896 62114 62201 62426 62692 62714 63175 63193 63195 63202 63210 63234
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 63287 63290 63291 63292 63294 64270 64326 65000 65097 65109 65111 65134
 65150 65174 65179 65194 65205 65206 65209 65210 65211 65223 65224 65225
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 89898 89913 89918 89922 89924 89927 89934 89935 89938 89940 90337 90380
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 101492 101612 103508 105720 105746 105803 105808 105896 105901 105904
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 114345 114694 114710 116134 116238 116439 116567 116901 116930 116954
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 125159 127532 127534 127846 128299 128301 128383 129130 129133 129134
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 133566 133853 133862 134251 134269 134329 134419 134720 134749 134957
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175471 177655 178268 178291 178387 179133 179190 179217 179233 179244
179902 180265 180274 181278 181279 181648 181665 181751 181773 182121
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182625 182627 182647 182666 182669 182675 182746 183309 183358 184274
184276 184280 184284 184306 184829 184937 184975 184976 184977 184978
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185487 185489 185490 185492 185666 185693 185906 185919 186106 186125
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187479 187607 187655 187657 187659 187692 187724 187755 188006 188020
188026 188053 188060 188234 188247 188257 188574 188835 188836 188837
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188966 188976 188986 189004 189214 189215 189310 189377 189386 189493
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 135496 136807 138699 138739 138816 138854 140348 140956 141109 141127
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